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ON THE DIMENSION AND THE INDEX OF THE SOLUTION SET OF NONLINEAR EQUATIONS

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ABSTRACT. We study the covering dimension and the index of the solution set to multiparameter nonlinear and semilinear operator equations involving Fredholm maps of positive index. The classes of maps under consideration are (pseudo) A-proper and either approximation-essential or equivariant approximation-essential. Applications are given to semilinear elliptic BVP's.

INTRODUCTION

It is the purpose of this paper to study the covering dimension and the index of the solution set of (equivariant) operator equations of the form

$$(1) \qquad H(\lambda, x) = f \qquad ((\lambda, x) \in D \subset R^m \times X, f \in Y)$$

as well as of semilinear operator equations of the form

$$(2) \qquad Ax + N(\lambda, x) = f \qquad ((\lambda, x) \in D \subset R^m \times D(A) \subset R^m \times X, \ f \in Y)$$

by developing various continuation principles involving A-proper homotopies. Here, X and Y are Banach spaces with a scheme $\Gamma_i = \{X_n, Y_n, Q_n\}$, $D \subset R^m \times X$ is an open subset, $A : D(A) \subset X \to Y$ is a linear Fredholm map of index $i(A) \geq 0$, and $N : \overline{D} \to Y$ is a suitable nonlinear multiparameter map. These continuation principles are extensions to the multiparameter case of the earlier ones in [Mi-2–8].

The structure and the covering dimension of the branches of solutions to these equations have been studied by many authors ([AA-1], [MP], [FMP-1,2]) and, when the parameter space is infinite dimensional, by [AA-2], [AMP] using cohomology theories (in particular, Čech cohomology). Recently, a simpler method, based on the notion of essential maps, has been used to study these problems in [IMPV] and, for equivariant maps, in [IMV-1] by developing a degree theory for such maps. For an excellent survey, see Ize [I].

Our study is based on the approximation-essential mapping approach as developed in [IMPV] and [Mi-7], and the basic dimension result of [IMV-1] for G-equivariant maps for some compact Lie group and of [FMP-1]. The index

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The majority of the results in the paper have been announced in [Mi-6].

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results are based on a finite dimensional index theory. As applications of the obtained results, we prove a number of dimension and index results for semilinear equations of the form

$$(3) \qquad Ax + Nx = f \qquad (x \in D(A), f \in Y)$$

where A is a Fredholm map of positive index and $A+N$ is A-proper. We still get the solvability results if the involved maps are only pseudo A-proper. Applications are given to semilinear elliptic boundary value problems.

1. Continuation theory for equivariant A-essential maps

Many problems in Applied Mathematics have symmetries. For example, the constitutive equations of continuum mechanics must be independent of the reference systems; rotations and translations cannot change the nature of the problem. A change in the origin in time should not change the nature of the equations in evolution problems (see [I] for more details). In this section we shall develop some continuation theory for approximation-essential maps dealing with the structure and covering dimension and with the index of the solution set of nonlinear operator equations. We assume that the maps have some symmetry property; i.e. they are equivariant relative to some compact Lie group G. Let $GL(X)$ be the space of all linear continuous isomorphisms of X equipped with the operator norm. Suppose that X is a Banach G-space or a representation of G; i.e., there is given a continuous homomorphism $T_g : G \to GL(X)$. We can (and will) assume that $T_g : X \to X$ is an isometry for each $g \in G$. If G acts on Banach spaces X and Y via $\{T_g\}$ and $\{\widetilde{T}_g\}$, respectively, then a map $T : X \to Y$ is said to be G-equivariant if $T(T_g x) = \widetilde{T}_g(Tx)$ for each $x \in X$ and $g \in G$.

For a subgroup H of G, we denote by $\mathrm{Fix}_X(H) = \{x \in X \mid T_h x = x$ for each $h \in H\}$. A subgroup $G_x = \{g \in G \mid T_g x = x\}$ is called the isotropy subgroup of G at x. Define $X_H = \{x \in X \mid G_x = H\}$ and $X_{(H)} = GX_H = \{x \in X \mid G_x = g^{-1}Hg$ for some $g \in G\}$. If G is a finite group, we define for any irreducible representation $V \subset X$ the number $\mu(V)$ as the greatest common divisor of the numbers $|G/H_i|$, where H_i ranges over all subgroups such that $V_{H_i} = \{x \in V \mid G_x = g^{-1}H_i g$ for some $g \in G\} \neq \varnothing$. If $V_1, \ldots, V_k$ are all irreducible representations contained in X, then $\mu(X)$ is defined as the greatest common divisor of the numbers $\mu(V_i)$. A subset $S \subset X$ is said to be G-invariant if $GS \subset S$. Let X/G be the orbit space of X relative to G.

A. **Dimension results.** Let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of Banach G-spaces X and Y, respectively, with $\mathrm{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$ and let $Q_n : Y \to Y_n$ be a linear projection onto Y_n with $\delta = \sup \|Q_n\| < \infty$. Suppose that $\dim X_n - \dim Y_n = i \geq 0$ for each n and some fixed i. We say that a scheme $\Gamma_i = \{X_n, Y_n, Q_n\}$ for (X, Y) is G-invariant if all X_n and Y_n are G-invariant. Let $T_n = Q_n T$.

Let $U \subset X$ be an open, not necessarily bounded, subset and S be an arbitrary subset of X. To define the class of admissible (pseudo) A-proper maps on $S \cap U$, we recall ([IMPV], [IMV-1]).

Definition 1.1. Let $S \subset X$ be a G-invariant subset. A map $T : U \to Y$ is said to be *admissible w.r.t.* a G-invariant Γ_i on $S \cap U$ if there is an open, bounded, and G-invariant set V_0 such that $T^{-1}(0) \cap S \subset V_0 \subset \overline{V}_0 \subset U$ and $(T_n)^{-1}(0) \cap S \subset V_0 \cap X_n$ for all n. If $T : \overline{U} \to Y$ and $G = \{e\}$, then it is admissible w.r.t. Γ_i on $S \cap \overline{U}$ if $T^{-1}(0) \cap S \subset V_0 \subset U$ for some open and bounded subset V_0 and $(T_n)^{-1}(0) \cap S \subset V_0 \cap X_n$ for all n.

Definition 1.2. An admissible map T w.r.t. Γ_i on $S \cap U$ (resp., $S \cap \overline{U}$) is said to be *Approximation-essential* (*A-essential* for short) w.r.t. Γ_i on $S \cap U$ (resp., $S \cap \overline{U}$) if, for each n, $T_n = Q_n T : U \cap X_n \to Y_n$ is continuous, and for any open, bounded, and G-invariant set V such that $(T_n)^{-1}(0) \cap S \subset V \cap X_n \subset \overline{V} \cap X_n \subset U \cap X_n$ (resp., $(T_n)^{-1}(0) \cap S \subset V \cap X_n \subset U \cap X_n$), any continuous G-equivariant extension $\widetilde{T}_n : \overline{V} \cap X_n \to Y_n$ of $T_n \,|\, \partial V \cap X_n \to Y_n$ has a zero in $S \cap V \cap X_n$.

Remark 1.1. If $G = \{e\}$, U is bounded, and $S \cap \overline{U}$ is closed, then $T : \overline{U} \to Y$ is A-essential w.r.t. Γ_i on $S \cap \overline{U}$ if and only if, for each n, the restriction $T_n \,|\, S \cap \partial U \cap X_n$ is essential with respect to $S \cap \overline{U} \cap X_n$ in the classical sense (cf. Dugundji-Granas [DG]); i.e., any continuous extension $\widetilde{T}_n : S \cap \overline{U} \cap X_n \to Y_n$ has a zero in $S \cap \overline{U} \cap X_n$. Hence, we see that in this case the A-essentiality w.r.t. Γ_i on $S \cap \overline{U}$ reduces to the one introduced by the author ([Mi-7]).

Definition 1.3. A map $T : U \to Y$ is (*pseudo*) *A-proper* w.r.t. Γ_i on $S \cap U$ if whenever V is an open and bounded set with $V \subset \overline{V} \subset U$ and $\{x_{n_k} \in S \cap V \cap X_{n_k}\}$ is such that $Q_{n_k} T x_{n_k} - Q_{n_k} f \to 0$ for some $f \in Y$, then $\{x_{n_k}\}$ has a subsequence converging to $x \in S \cap U$ (there is $x \in S \cap U$) with $Tx = f$. If $T : \overline{U} \to Y$, then it is (pseudo) A-proper w.r.t. Γ_i on $S \cap \overline{U}$ if it has the above property whenever V is open, bounded, and $V \subset U$.

Definition 1.4. A homotopy $H : [0, 1] \times U \to Y$ is *admissible* w.r.t. a G-invariant Γ_i on $S \cap U$ if there is an open, bounded and G-invariant subset V_0 such that for each $t \in [0, 1]$, $H_t = H(t, \cdot)$, $H_t^{-1}(0) \cap S \subset V_0 \subset \overline{V}_0 \subset U$ and $(Q_n H_t)^{-1}(0) \cap S \subset V_0 \cap X_n$ for all n. If $H : [0, 1] \times \overline{U} \to Y$, then it is clear how to define its admissibility w.r.t. Γ_i on $S \cap \overline{U}$.

Note that the admissibility of H_t means that, for each $t \in [0, 1]$, H_t and $Q_n H_t$ have no zero near the boundaries ∂U and $\partial U \cap X_n$.

Definition 1.5. A homotopy $H : [0, 1] \times U \to Y$ is *A-proper* w.r.t. Γ_i on $S \cap U$ if $Q_n H_t : U \cap X_n \to Y_n$ is continuous for each t and n, and if $\{x_{n_k} \in S \cap V \cap X_{n_k}\}$, with V open and bounded such that $V \subset \overline{V} \subset U$, $t_k \in [0, 1]$ with $t_k \to t$ and $Q_{n_k} H(t_k, x_{n_k}) - Q_{n_k} f \to 0$ as $k \to \infty$ for some $f \in Y$, then a subsequence of $\{x_{n_k}\}$ converges to $x \in S \cap U$ and $H(t, x) = f$. Similarly, we define the A-properness of $H : [0, 1] \times \overline{U} \to Y$.

The classes of A-proper and pseudo A-proper maps are very general and we refer to [FMP-1, Mi-1–8, Pe-1–2] for many examples of such maps. We note also that (pseudo) A-proper maps w.r.t. Γ_i with $i > 0$ have been first studied by the author in [Mi-4, 6] and, independently, by Fitzpatrick-Massabo-Pejsachowicz (cf. [FMP-1]). For a recent survey of the (pseudo) A-proper mapping theory, we refer to [Mi-9].

We have the following transversality result.

Theorem 1.1. *Let D be U or $\overline{U}$ and $H : [0, 1] \times D \to Y$ be an admissible and A-proper homotopy w.r.t. Γ_i on $S \cap D$. Then H_1 is A-essential w.r.t. Γ_i on $S \cap D$ if and only if H_0 is such.*

Proof. When U is bounded and $S \cap \overline{U}$ is closed, the theorem was proved by the author [Mi-7]. Using similar arguments, it is easy to prove it in this generality (cf. also [IMPV]). □

Definition 1.6. A map $T : U \to Y$ is said to be *sectionally proper* on a closed (in X) subset $C \subset U$ if and only if for any finite dimensional subspace Y_n of Y and any compact set $K \subset Y_n$, the set $T^{-1}(K)$ is compact.

Recall that, if K is a topological space and m is a positive integer, then K is said to have covering dimension equal to m provided that m is the smallest integer such that whenever $\mathcal{F}$ is a family of open subsets of K, whose union covers K, then there is a refinement, $\mathcal{F}_1$, of $\mathcal{F}$ whose union also covers K, and no subfamily of $\mathcal{F}_1$ consisting of more than $m + 1$ members has nonempty intersection. If K fails to have this refinement property for each positive integer, then K is said to have infinite dimension. When $x \in K$, then K is said to have dimension (covering) at least m at x if each neighborhood of x, in A, has dimension at least m. In the absence of a manifold structure on K, the concept of dimension is the natural way to describe its size.

We need the following covering dimension result for G-equivariant A-proper homotopies of Ize-Massabo-Vignoli [IMV-1].

Theorem 1.2. *Let S be closed in U and $H : [0, 1] \times U \to Y$ be a sectionally proper on bounded and closed subsets of $S \cap U$ admissible G-equivariant A-proper homotopy w.r.t. G-invariant scheme $\Gamma_i = \{\Lambda \times R^m \times X_n, Y_n \times R^k, Q_n\}$ on $S \cap U$. Let H_0 be A-essential w.r.t. Γ_i on $S \cap U$. Then there exists an invariant set $\Sigma \subset S \cap U$ which is minimal, closed (in U), and*

(a) *H_1 is A-essential w.r.t. Γ_i on $\Sigma \cap U = \Sigma$ and so $H_1^{-1}(0) \cap \Sigma \neq \varnothing$.*

(b) *If $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are proper, closed, and invariant subsets with $\Sigma_1 \cap \Sigma_2 = \varnothing$, then either $\Sigma_1 = \varnothing$ or $\Sigma_2 = \varnothing$. This is equivalent to saying that Σ/G is connected. If G is connected, then so is Σ.*

(c) *If $\mathrm{Fix}_Y(G) \neq \{0\}$, then Σ is either unbounded or $\overline{\Sigma} \cap \partial U \neq \varnothing$. Assume that $Y = \mathrm{Fix}_Y(G) \oplus Y_2$, where Y_2 is such that $\mathrm{Fix}_{Y_2}(G) = \{0\}$, and decompose H_1 as $H_1 = (h_1, h_2)$, where $h_1 : U \to Y_1 = \mathrm{Fix}_Y(G)$ and $h_2 : U \to Y_2$. Then there exists an invariant minimal subset $\widetilde{\Sigma}$ contained in $h_2^{-1}(0) \cap \Sigma$ such that*

(i) *h_1 is A-essential and invariant on $\widetilde{\Sigma} \cap U$ w.r.t. Γ_m and, in particular, $\widetilde{\Sigma}$ is either unbounded, or $\mathrm{cl}(\widetilde{\Sigma}) \cap \partial U \neq \varnothing$ (provided $\mathrm{Fix}_Y(G) \neq \{0\}$).*

(ii) *If $\widetilde{\Sigma}/G = \Sigma_1 \cup \Sigma_2$ with Σ_1, Σ_2 closed and proper subsets of $\widetilde{\Sigma}/G$, then $\dim(\Sigma_1 \cap \Sigma_2 \cap X_n) \geq \dim \mathrm{Fix}_{Y_n}(G) - 1$ for infinitely many n. In particular, $\widetilde{\Sigma}/G$ is connected and has infinite dimension at each point.*

When $G = \{e\}$, then every map is G-equivariant. Hence, as a particular case of Theorem 1.2 we have the covering dimension result for A-proper homotopies due to Ize-Massabo-Pejsachowicz-Vignoli [IMPV] and to Fitzpatrick-Massabo-Pejsachowicz [FMP-1] in a less general form.

The following continuation theorem gives some conditions on H when Theorem 1.2 can be applied .

Theorem 1.3. *Let U be an open subset of X, S be closed in U, and $H : [0, 1] \times U \to Y$ be a sectionally proper on bounded and closed subsets of $S \cap U$ A-proper homotopy w.r.t. Γ_i on $S \cap U$. Suppose that there is an open and bounded subset D of X such that*

(i) $H(t, x) \neq f$ *for* $x \in \partial D \cap S$, $t \in [0, 1]$.

(ii) $H(0, x) \neq tf$ *for* $x \in \partial D \cap S$, $t \in [0, 1]$.

(iii) H_0 *is A-essential w.r.t.* Γ_i *on* $S \cap U$.

Then the conclusions of Theorem 1.2 *hold for* $H_1 - f$ *with* $G = \{e\}$.

Proof. Since H_t is sectionally proper on closed and bounded subsets of $S \cap U$ and $H_t^{-1}(f) \cap S \cap \overline{D}$ is such a set, then $H_t^{-1}(f) \cap S \cap \overline{D}$ is a compact set contained in D by (ii) for each $t \in [0, 1]$. Hence, for each $t \in [0, 1]$, there is an open and bounded subset V_t of X such that $H_t^{-1}(f) \cap S \subset V_t \subset \overline{V}_t \subset D$. Define $V = \bigcup_{t \in [0, 1]} V_t$. Then V is open and bounded and $H_t^{-1}(f) \cap S \subset V \subset D$ for each $t \in [0, 1]$; i.e., H_t is admissible on $S \cap \overline{D}$.

Next, set $F_t = H_t - f$. Then $F_t^{-1}(0) = H_t^{-1}(f)$ and F_t is admissible on $S \cap \overline{D}$. Since H_t is an A-proper homotopy, then arguing by contradiction and using the admissibility of H_t we get that F_t is admissible w.r.t. Γ_i on $S \cap \overline{D}$. Next, set $G_t = H_0 - tf$. Then, (ii) implies that $G_t^{-1}(0) \cap S = H_0^{-1}(tf) \cap S$ is compact and contained in D for each $t \in [0, 1]$. Hence, there is an open and bounded set W_t such that $G_t^{-1}(0) \cap S \subset W_t \subset \overline{W}_t \subset D$. Then $W = \bigcup_{t \in [0, 1]} W_t$ is open and bounded and, as above, we get that G_t is admissible w.r.t. Γ_i on $S \cap D$.

Finally, set $W_0 = V \cup W$. Then W_0 is open and bounded with $W_0 \subset D$ and G_t is an admissible and A-proper homotopy w.r.t. Γ_i on $S \cap \overline{D}$. Since $G_0 = H_0$ is A-essential w.r.t. Γ_i on $S \cap \overline{D}$, so is $G_1 = F_0$. Hence, since F_t is also an admissible and A-proper homotopy w.r.t. Γ_i on $S \cap \overline{D}$, it follows that $F_1 = H_1 - f$ is A-essential w.r.t. Γ_i on $S \cap \overline{D}$. Therefore, the conclusions of the theorem follow from Theorem 1.2 with $G = \{e\}$. □

When H_1 is just pseudo A-proper, we can still get the solvability of $H(1, x) = f$. Let $V \subset X$ be a dense subspace, $D \subset R^m \times X$ be an open subset. We say that $H : [0, 1] \times \overline{D} \cap (R^m \times V) \to Y$ is an m-parameter A-proper homotopy w.r.t. $\Gamma_m = \{R^m \times X_n, Y_n; Q_n\}$ for $(R^m \times X, Y)$ if whenever $\{(\lambda_{n_k}, x_{n_k}) \mid (\lambda_{n_k}, x_{n_k}) \in \overline{D} \cap R^m \times X_{n_k}\}$ is bounded and $t_k \in [0, 1]$ such that $\lambda_{n_k} \to \lambda$ and $t_k \to t$ and $Q_{n_k} H(t_k, \lambda_{n_k}, x_{n_k}) \to f$, then a subsequence $x_{n_{k(i)}} \to x$ and $H(t, \lambda, x) = f$.

Theorem 1.4. *Let $D \subset R^m \times X$ be open and bounded, $f \in Y$, and $H : [0, 1] \times \overline{D} \cap (R^m \times V) \to Y$ be an A-proper homotopy on $[0, \varepsilon] \times \partial D \cap (R^m \times V)$ w.r.t. $\Gamma_m = \{R^m \times X_n, Y_n, Q_n\}$ for each $\varepsilon \in (0, 1)$. Let $H(1, \cdot, \cdot)$ be pseudo A-proper w.r.t. Γ_m, $H(t, \lambda, x)$ be continuous at 1 uniformly at $(\lambda, x) \in \overline{D} \cap (R^m \times V)$, and*

(i) $H(t, \lambda, x) \neq f$ *for* $(\lambda, x) \in \partial D \cap (R^m \times V)$, $t \in [0, 1)$.

(ii) $H(0, \lambda, x) \neq tf$ *for* $(\lambda, x) \in \partial D \cap (R^m \times V)$, $t \in [0, 1]$.

(iii) H_0 *is A-essential w.r.t.* Γ_m *on* $\overline{D}$.

Then the equation $H(1, \lambda, x) = f$ *is solvable.*

Proof. Set $H_t = H(t, \cdot, \cdot)$. Arguing by contradiction, it is easy to see that the A-properness of H_0 and (ii) imply that there is an $n_1 \geq 1$ such that

$$(4) \qquad Q_n H(0, \lambda, x) \neq t Q_n f \quad \text{for } x \in \partial D \cap (R^m \times V), \ t \in [0, 1], \ n \geq n_1.$$

Hence, $H_0 - f$ is A-essential on $\overline{D}$.

Now, let $\varepsilon \in (0, 1)$ be fixed. Then, arguing again by contradiction, we see that the A-properness of H on $[0, \varepsilon] \times \partial D \cap (R^m \times V)$ and (i) imply that there is an $n_2 = n_2(\varepsilon) \geq n_1$ such that for each $n \geq n_2$

$$(5) \qquad Q_n H(t, \lambda, x) \neq Q_n f \quad \text{for } (\lambda, x) \in \partial D \cap (R^m \times X_n), \ t \in [0, \varepsilon],$$

with $n_1(\varepsilon_1) \geq n_2(\varepsilon_2)$ whenever $\varepsilon_1 > \varepsilon_2$. Using this and the homotopy $F_n : [0, 1] \times \overline{D} \cap (R^m \times X_n) \to Y_n$, given by $F_n(t, \lambda, x) = Q_n H(\varepsilon t, \lambda, x) - Q_n f$, we get that $H_\varepsilon - f$ is A-essential on $\overline{D}$. Therefore, for each such $n \geq n_2$ there is an $(\lambda_n, x_n) \in D \cap (R^m \times X_n)$ such that $Q_n H_n(\varepsilon, \lambda_n, x_n) = Q_n f$.

Next, let $\varepsilon_k \in (0, 1)$ be increasing and $\varepsilon_k \to 1$ and $(\lambda_{n_k}, x_{n_k}) \in D \cap (R^m \times X_{n_k})$ be such that $Q_{n_k} H_{n_k}(\varepsilon_k, \lambda_{n_k}, x_{n_k}) = Q_{n_k} f$ for $k \geq 1$. By the continuity of H_t at 1 uniformly for (λ, x), we get that $Q_{n_k} H(1, \lambda_{n_k}, x_{n_k}) \to f$, and $H(1, \lambda, x) = f$ for some (λ, x) by the pseudo A-properness of H_1. □

Next, we prove an extension of the first Fredholm theorem to nonlinear pseudo A-proper maps. We say that a map $A : V \subset X \to Y$ is α-positively homogeneous if $A(tx) = t^\alpha A(x)$ for all $x \in V$, $t > 0$, and some $\alpha > 0$.

Theorem 1.5. *Let V be a dense subspace of X and $A : V \to Y$ be a positively α-homogeneous for some $\alpha > 0$ and A-essential A-proper map w.r.t. $\Gamma_i = \{X_n, Y_n, Q_n\}$ on V. Suppose that $Ax = 0$ implies $x = 0$ and $N : X \to Y$ is quasibounded; i.e.,*

$$|N| = \limsup_{\|x\| \to \infty} \|Nx\| / \|x\|^\alpha < \infty$$

and $|N|$ is sufficiently small. Then, if $A + N$ is pseudo A-proper w.r.t. Γ_i, the equation $Ax + Nx = f$ is solvable for each $f \in Y$.

Proof. By Lemma 2.1 in [Mi-1] there is a $c > 0$ and an $n_0 \geq 1$ such that $\|Q_n Ax\| \geq c\|x\|^\alpha$ for each $x \in X_n$ and $n \geq n_0$. Let $H_f(t, x) = Ax + tNx - tf$ for each $f \in X$. Then, arguing by contradiction, it is easy to show that there exists an $r_f > 0$ such that $Q_n H_f(t, x) \neq 0$ for all $x \in \partial B(0, r_f) \cap X_n$, $t \in [0, 1]$, and $n \geq n_0$. Since $H_f(0, \cdot) = A$ is A-essential, by Theorem 1.1 $H_f(1, \cdot)$ is A-essential w.r.t. Γ_i on $S \cap \overline{U}$. Hence, there is an $x_n \in S \cap \overline{U} \cap X_n$ such that $Q_n H(1, x_n) = 0$ for each $n \geq n_0$. Since $H_f(1, \cdot)$ is pseudo A-proper, there is an $x \in S \cap \overline{U}$ such that $Ax + Nx = f$. □

Special cases. Now, we shall derive several special cases of the above results using either the G-degree, or complementing maps or the homotopy degree. The G-degree has been defined, studied, and applied by many authors for various groups G (Dancer, Ize, Massabo, Vignoli, Geba and others, see [IMV-1,2], [IV], [GKW] and [I] for references). The G-degree has been developed in [IMV-1,2] for general G and is used below.

We have the following continuation result.

Theorem 1.6. (a) *Let D be an open bounded G-invariant subset of $\Lambda \times R^m \times X$ and $H : \overline{D} \to Y \times R^k$ be a G-equivariant and A-proper homotopy w.r.t. a G-invariant scheme $\Gamma_i = \{\Lambda \times R^m \times X_n, Y_n \times R^k, Q_n\}$. Suppose that for some $\lambda_0 \in \mathrm{Fix}_\Lambda(G)$, $Q_n H(\lambda_0, x) \neq 0$ on $\partial D_{\lambda_0} \cap (\Lambda \times R^m \times X_n) = \partial\{(\lambda_0, x) \in D\} \cap (\Lambda \times R^m \times X_n)$ and $\deg_G(Q_n H_{\lambda_0}, D_{\lambda_0} \cap (\Lambda \times R^m \times X_n), 0) \neq 0$*

for all large n. *Assume that the* G*-equivariant Freudenthal suspension theorem applies and that* $\Lambda = \mathrm{Fix}_{\Lambda}(G) \oplus \Lambda_1$ *with* $\dim \mathrm{Fix}_{\Lambda}(G) > 0$. *Then there exists a "continuum"* C *of solutions of the equation* $H(\lambda, x) = 0$ *with* $\lambda \in \mathrm{Fix}_{\Lambda}(G)$, *such that* $\overline{C} \cap \partial D \neq \varnothing$ *and* C/G *is connected and has dimension at each point at least* $\dim \mathrm{Fix}_{\Lambda}(G)$.

(b) [IMPV] *let* V *be a dense subspace of* X, $B_1(0, R_1) \subset R^m$, $B_2(0, R_2) \subset X$ *be open balls, and let* $H : [0, 1] \times \overline{B}_1 \times (\overline{B}_2 \cap V) \to Y$ *be an* A*-proper homotopy w.r.t.* $\Gamma_m = \{R^m \times X_n, Y_n, Q_n\}$ *with* $X_n \subset X$ *and such that each* H_t *is sectionally proper on bounded and closed sets. Suppose that* $Q_n H_0(0, x) \neq 0$ *on* ∂B_2 *and the* m*th suspension of* $Q_n H_0 \mid \partial B_2 \cap X_n$ *is nontrivial for each large* n *and for* $D = B_1 \times B_2$, *conditions* (i)–(ii) *of Theorem* 1.4 *hold.*

Then there is a minimal connected subset Σ *of* $S = H_1^{-1}(f)$ *that has dimension at least* m *at each point.*

Proof. (a) Define the map $F(\lambda, x) = (H(\lambda, x), \lambda - \lambda_0)$ and note that $Q_n F(\lambda, x) \neq 0$ on ∂D. Hence,

$$\begin{aligned} &\deg_G(\widehat{Q}_n F, D \cap (\Lambda \times R^m \times X_n), 0) \\ &\quad = \Sigma^G \deg_G(Q_n H_{\lambda_0}, D_{\lambda_0} \cap (\Lambda \times R^m \times X_n), 0) \neq 0. \end{aligned}$$

Thus, the map $\widehat{Q}_n F$ is G-epi on D and then the map $\lambda - \lambda_0$ is G-epi on the set of zeros of the equation $H(\lambda, x) = 0$ with $\lambda \in \mathrm{Fix}_{\Lambda}(G)$ (cf. [IMV-1]). Now, the conclusions follow from Theorem 1.2 since the notion of a G-epi map is more general than that of the G-degree ([IMV-2]). □

Let V be a dense subspace of X and $D \subset R^m \times X$ be open and bounded. Recall that [FMP-1] a continuous map $C : \overline{D} \to R^m$ is called a complement of $T : \overline{D} \cap (R^m \times V) \to Y$ provided that $\deg(\widehat{Q}_n(T, C), D \cap X_n, 0) \neq 0$ for all large n, where $(T, C)(\lambda, v) = (T(\lambda, v), C(\lambda, v)) \in Y \oplus R^m$ is A-proper w.r.t. $\widehat{\Gamma} = \{R^m \times V_n, Y_n \times R^m, \widehat{Q}_n\}$ with $\widehat{Q}_n(y, \lambda) = (Q_n y, \lambda)$ and $V_n = X_n \cap V$. When D is unbounded but $(T, C)^{-1}(0)$ is bounded, we define

$$\deg(\widetilde{Q}_n(T, C), D \cap X_n, 0) = \deg(\widetilde{Q}_n(T, C), U \cap X_n, 0),$$

where U is any bounded neighborhood of $(T, C)^{-1}(0)$ in D.

If H_0 is a complementing map, then it is A-essential and Theorem 1.3 holds for such homotopies.

Next, when H_1 is just pseudo A-proper, we can still establish the solvability of $H(1, \lambda, x) = f$.

Theorem 1.7. *Let* V *be a dense subspace of* X, $D \subset R^m \times X$ *be open and bounded,* $f \in Y$, *and* $H : [0, 1] \times \overline{D} \cap (R^m \times V) \to Y$. *Suppose that* H *is an* A*-proper homotopy on* $[0, \varepsilon] \times \partial D \cap (R^m \times V)$ *w.r.t.* $\Gamma_m = \{R^m \times X_n, Y_n, Q_n\}$ *for each* $\varepsilon \in (0, 1)$, $H(1, \cdot, \cdot)$ *is pseudo* A*-proper w.r.t.* Γ_m; $H(t, \lambda, x)$ *is continuous at* 1 *uniformly for* $(\lambda, x) \in \overline{D} \cap (R^m \times V)$. *Let conditions* (i)–(ii) *of Theorem* 1.4 *hold as does either one of the following conditions*:

(iii) H_0 *is complemented by* $C : \overline{D} \to R^m$.

(iv) H *is* G*-equivariant,* Γ_m *is* G*-invariant, and*

$$\deg_G(Q_n H_0, D \cap (R^m \times X_n), 0) \neq 0.$$

(v) $D = B(0, R)$ *and* $Q_nH_0 : \partial B(0, R) \cap (R^m \times X_n) \to Y_n \backslash \{0\}$ *has the nontrivial stable homotopy for all large* n.
Then the equation $H(1, \lambda, x) = f$ *is solvable in* $R^m \times V$.

Proof. In either case we have that H_0 is A-essential w.r.t. Γ_m and Theorem 1.4 applies. □

Our next result is the following continuation theorem for pseudo A-proper G-equivariant maps with G satisfying

(G) G is either finite with $\mu(X) > 1$ or is an infinite compact Lie group such that $\mathrm{Fix}(K) = 0$ for some subtorus K of G.

Theorem 1.8. *Let* $R^m \times X$ *be a Banach G-space,* R^m *be G-invariant,* $\Gamma_0 = \{X_n, P_n\}$ *be a G-invariant scheme for* X, *and condition* (G) *hold on* $R^m \times X$.

(a) *If* $m = 0$ *and* $H : [0, 1] \times \overline{B}_r \to X$, *then* $H(1, x) = 0$ *is solvable if*

(i) $H_1 = H(1, \cdot)$ *is pseudo A-proper at* 0 *w.r.t.* Γ;

(ii) $P_nH(t, x) \neq 0$ *for all* $x \in \partial B_r \cap X_n$, $t \in [0, 1]$, $n \geq n_0 \geq 1$;

(iii) $H_0 : \overline{B}_r \to X$ *is G-equivariant.*

(b) *If* $m > 0$, $T : R^m \times X \to X$ *is G-equivariant and pseudo A-proper at* 0 *w.r.t.* $\Gamma_m = \{R^m \times X_n, X_n, P_n\}$, *then the equation* $T(\lambda, x) = 0$ *is solvable. If* T *is also A-proper at* 0 *w.r.t.* Γ_m, *then*

$$Z_r = \{(\lambda, x) \mid \|(\lambda, x)\| = r,\ T(\lambda, x) = 0\} \neq \varnothing$$

for each $r > 0$.

Proof. (a) Let G be finite and fix $n \geq n_0$. Since $P_nH_0 : \overline{B}_r \cap X_n \to X_n$ is G-equivariant and $P_nH_0(x) \neq 0$ for $x \in \partial B_r \cap X_n$, we know that [I]

$$\deg(P_nH_1, B_r \cap X_n, 0) = \deg(P_nH_0, B_r \cap X_n, 0) = 1 + k|G|.$$

If G is infinite, since $X_n \subset X$, we have that

$$\mathrm{Fix}(K) \cap X_n = 0, \quad \text{and} \quad \deg(P_nH_1, B_r \cap X_n, 0) = 1$$

for each $n \geq n_0$. Hence, there is an $x \in B_r \cap X_n$ such that $P_nH_1(x_n) = 0$. By the pseudo A-properness of H_1, the equation $H(1, x) = 0$ is solvable in $\overline{B}_r$.

(b) Let $r > 0$ be fixed and suppose that for some n, $P_nT : \partial B_r \cap (R^m \times X_n) \to X_n \backslash \{0\}$. Let $\overline{T}_n : \partial B_1 \cap (R^m \times X_n) \to \partial B_1 \cap X_n$ be given by $\overline{T}_n(x) = P_nT(rx)/\|P_nT(rx)\|$ and $i_n : \partial B_1 \cap R^m \to \partial B_1 \cap (R^m \oplus X_n)$ be the natural inclusion. Then $\overline{T}_n$, i_n, and $\widehat{T}_n = i_n\overline{T}_n : \partial B_1 \cap (R^m \times X_n) \to \partial B_1 \cap (R^m \times X_n)$ are G-equivariant maps, and $\deg(\widehat{T}_n) \neq 0$. But, since $\widehat{T}_n$ factors through $\partial B_1 \cap R^m \subset \partial B_1 \cap (R^m \oplus X_n)$ and $R^m \neq 0$, we have that $\deg(\widehat{T}_n) = 0$. This contradiction shows that $\{x \in \partial B_r \cap (T^m \times X_n) \mid P_nTx = 0\} \neq \varnothing$ for each n. Then the conclusions follow from the (pseudo) A-properness of T. □

Now, regarding the schemes used above, the following result is useful.

Proposition 1.1. *If* X *is a Banach G-space and separable* (π_k*-space, respectively*), *then there are finite dimensional G-invariant subspaces* X_n *of* X (*with* $X_1 \subset X_2 \subset \cdots$, *respectively*) *such that* $\mathrm{dist}(x, X_n) \to 0$ *as* $n \to \infty$ *for each* $x \in X$.

Proof. Let U_n be a sequence of finite dimensional subspaces of X such that $\mathrm{dist}(x, U_n) \to 0$ as $n \to \infty$ for each $x \in X$. Define the linear subspaces of

X by $X_n = GU_n = \{T_g x \mid x \in U_n, g \in G\}$. If $S(U_n)$ is the unit sphere in U_n, then $S(X_n) = GS(U_n)$ is the unit sphere in X_n since G acts by isometries on X. Since $S(X_n)$ is a compact subset of X_n as a continuous image of the compact set $G \times S(U_n)$, it follows that X_n is finite dimensional. Moreover, each X_n is G-invariant and $\mathrm{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$ since each $T_g : X \to X$ is an isometry. Finally, if $U_1 \subset U_2 \subset \cdots$, then $X_1 \subset X_2 \subset \cdots$. □

B. **On the index of the solution set.** Our next step is to connect a given representation of G on X with an index theory and then estimate the index of the solution set Z_r from below. Moreover, for some concrete index theories one can then estimate the covering dimension of Z_r from below.

By an index theory I on a Banach space X we mean a triplet $\{\Sigma, M, i\}$, where Σ is a family of closed subsets of X such that $A \cup B$, $A \cap B$, and $\overline{A \backslash B} \in \Sigma$ for all $A, B \in \Sigma$, M is a set of continuous maps containing the identity and is closed under composition with $T(\overline{A}) \in \Sigma$ for each $A \in \Sigma$ and each $T \in M$, and $i : \Sigma \to N \cup \{+\infty\}$ is a map having suitable properties ([FR, Be-1]). Let $\{T_g\}$ be a representation of G on X and define $\Sigma(T_g) = \{A \subset X \mid A$ is closed and G-invariant$\}$ and $M(T_g) = \{h \in C(X, X) \mid h$ is G-equivariant$\}$. An index theory $\{\Sigma, M, i\}$ is related to the representation of G on X if $\Sigma = \Sigma(T_g)$ and $M = M(T_g)$. It is said to have the d-dimension property if there is a positive integer d such that $i(X_{dk} \cap \partial U) = k$ for all dk-dimensional subspaces $X_{dk} \in \Sigma$ such that $X_{dk} \cap \mathrm{Fix}(G) = \varnothing$ and all closed bounded neighborhoods $U \in \Sigma$ of zero.

In the literature there are many index theories and we mention only the index theory induced by the genus function of Krasnoselskii ([K], [R-1]), the indices in [FR], [Be-1,2], [FHR], [FH], [M-2], [B-1], [LW], etc. (see the references in these works). Let $[r]$ denote the biggest integer $n \leq r$.

We need the following finite dimensional result.

Theorem 1.9. *Let X_n be a finite dimensional Banach G-space, X_k its proper G-invariant subspace,* $\mathrm{Fix}(G) = 0$, *and $\{\Sigma, M, i\}$ an index theory related to the G representation on X_n and such that $i(K) < \infty$ whenever $K \in \Sigma$ is compact. Suppose that Ω is a bounded closed neighborhood of* 0 *in X_n, $\partial\Omega$ is G-invariant, and $f : \Omega \to X_k$ is continuous and G-equivariant on $\partial\Omega$. Then, if $Z = \{x \in \partial\Omega \mid f(x) = 0\}$, $i(Z) \geq i(\partial\Omega) - i(S^1)$, where $S^1 = \partial B(0, 1) \subset X_k$.*

Theorem 1.10. (a) *Let X be a Banach G-space, $T : X \to X$ be G-equivariant and pseudo A-proper at* 0 *w.r.t. a G-invariant scheme $\Gamma_m = \{X_m = X_0 \oplus V_n, V_n, Q_n\}$ for X with* $\dim X_0 = m > 0$ *and condition* (G) *hold. Then the equation $Tx = 0$ is solvable and, if T is also A-proper at* 0 *w.r.t. Γ_m, $Z_r = \{x \in \partial B_r \mid Tx = 0\} \neq \varnothing$ for each $r > 0$.*

(b) *Let X and Y be Banach G-spaces,* $\mathrm{Fix}(G) = 0$ *in X, $K : X \to Y$ be a linear G-equivariant homeomorphism, and let $\{\Sigma, M, i\}$ be an index theory related to the G-representation on X and having the d-dimension property. Suppose that $\Omega \in \Sigma$ is a bounded closed neighborhood of* 0 *in X and $T : \partial\Omega \to Y$ is G-equivariant and A-proper at* 0 *w.r.t. a G-invariant scheme $\Gamma = \{X_n, Y_n = K(X_{n-m}), Q_n\}$ for (X, Y) with $m > 0$. Then, if $Z = \{x \in \partial\Omega \mid Tx = 0\}$, $i(Z) \geq [m/d]$.*

Proof. Part (a) follows from Theorem 1.8. Let $n \geq 1$ be fixed and $\Omega_n = \Omega \cap X_n$. Then $\partial\Omega \cap X_n$ is G-invariant and $Q_n T : \partial\Omega \subset X \to Y_n$ is G-equivariant.

Indeed, let $x \in \partial\Omega$ and $g \in G$ be fixed and $y_k \in Y_k$ be such that $y_k \to Tx$ in Y. Then

$$Q_n T(T_g x) = Q_n \overline{T}_g \left(\lim_k y_k\right) = \lim_k Q_n \overline{T}_g(y_k) = \overline{T}_g Q_n T x.$$

Moreover, $K^{-1}Q_n T : \partial\Omega_n \to X_{n-m}$ is G-equivariant since such is K^{-1}; i.e., $K^{-1}(\overline{T}_g y) = T_g(K^{-1}y)$ for each $y \in Y$, $g \in G$.

Let $Z_n = \{x \in \partial\Omega_n \mid K^{-1}Q_n Tx = 0\}$. By Theorem 1.9 we have that $i(Z_n) \geq i(\partial\Omega_n) - i(S^{n-m}) \geq [r(n)/d] - [r(n) - m/d]^+ = [m/d]$, where $r(n) = \dim X_n$, since $[r(n)/d] \leq k$ for some k with $r(n) = dk + l$ and $l \in [0, d)$ and $[(r(n) - m)/d]^+ \geq k_1$ for some k_1 with $r(n) - m = dk_1 + l_1$ and $l_1 \in [0, d)$.

Next, since T is continuous and A-proper at 0 w.r.t. a projectionally complete scheme, it follows that T is proper on bounded and closed subsets of X, and consequently Z is compact in X. Hence, $i(Z) < \infty$ and, for some δ-neighborhood of Z, $i(N_\delta(Z)) = i(Z)$. Moreover, there is an $n_0 \geq 1$ such that $Z_n \subset N_\delta(Z)$ for each $n \geq n_0$. If not, then there would exist $x_{n_k} \in Z_{n_k} \backslash N_\delta(Z)$ such that for each $k \geq 1$, $Q_{n_k} T x_{n_k} = 0$ and some subsequence $x_{n_{k(i)}} \to x \in \partial\Omega$ by the A-properness of T at 0 with $Tx = 0$. But, $x \in \overline{\partial\Omega \backslash N_\delta(Z)}$ since $\partial\Omega \cap X_n \backslash N_\delta(Z) \subset \partial\Omega \backslash N_\delta(Z)$ for each n, in contradiction to $x \in Z$. Hence, such an $n_0 \geq 1$ exists, and for $n \geq n_0$, $i(Z) \geq i(Z_n) \geq [r(n)/d] - [r(n) - m/d]^+ = [m/d]$. □

Now, let us look at some special cases. Let $G = Z_2 = \{1, -1\}$ and let its representation on a real Banach space X be given by: $T_1 x = x$ and $T_{-1}x = -x$ for all $x \in X$. If $A \in \Sigma(T_g) = \{$ closed subsets of X symmetric with respect to $0\}$, we define the genus of A, $\gamma(A) = k$, if k is the smallest integer such that there exists a continuous odd map $\phi : A \to R^k \backslash 0$. If such a map does not exist we set $\gamma(A) = \infty$ and set also $\gamma(\varnothing) = 0$. Then it is well known that $\{\Sigma(T_g), M(T_g), \gamma\}$ is an index theory which possesses the dimension property with $d = 1$ (cf. [K, R-1]) and $\dim(A) \geq \gamma(A) - 1$. In this case Theorem 1.10 was obtained by the author [Mi-5] and extends an earlier result of Holm and Spanier [HS] and Rabinowitz [R-2] for compact perturbations of Fredholm maps of positive index.

Next, consider the multiplicative group of complex numbers $G = S^1 = \{z \in C \mid |z| = 1\}$ and a unitary representation $\{T_z\}$ of this group on a real Hilbert space. For simplicity, we shall write T_s instead of T_z if $z = e^{is}$, $s \in [0, 2\pi)$. If $A \in \Sigma(T_s) = \{$ closed T_s-invariant subsets of $H\}$, we set (Fadell-Rabinowitz [FR], Benci [Be-1]) $\tau(A) = k$ if k is the smallest integer for which there exist a positive integer n and a continuous map $\phi : A \to C^k \backslash \{0\}$ such that $\phi(T_s x) = e^{ins}\phi(x)$ for all $x \in A$ and $s \in [0, 2\pi)$. If such a map does not exist, we set $\tau(A) = +\infty$ and set also $\tau(\varnothing) = 0$. If X_k is an invariant subspace of H with $X_k \cap \mathrm{Fix}(S^1) = \varnothing$, one can show that its dimension is even and ([FR], [Be-1]) that $\{\Sigma(T_s), M(T_s), \tau\}$ is an index theory having the dimension property with $d = 2$.

Finally, consider a normed linear space X over $F = R$ or C or the quaternions H, and let G be the unit sphere in F. Let G act freely on $X_* = X \backslash \{0\}$; i.e., $\mathrm{Fix}(G) = \{0\}$. For $A \in \Sigma(T_g) = \{$ closed G-invariant subsets of $X_*\}$, we let $\mathrm{Ind}_F(A)$ be the Fadell-Rabinowitz index of A [FR, p. 148]. Then $\{\Sigma(T_g), M(T_g), \mathrm{Ind}_F\}$ is an index theory such that $\mathrm{Ind}_F(A) \dim F \leq \dim A$

for each $A \in \Sigma(T_g)$ and, if $X = F^n$, the Euclidean n-space over F and $\overline{U} \in \Sigma(T_g)$ is a bounded neighborhood of 0 in X, then $\mathrm{Ind}_F\, \partial U = n$ (see [FR]). Thus, their index satisfies the dimension property in X over F with $d = 1$. For other examples, see [FHR], [FH], [M-2], [B-1], [LW].

Remark 1.2. The above results can be used to study the index of the solution set of the k-parameter eigenvalue problem with symmetries $Ax + N(\lambda, x) = 0$ where $A : X \to Y$ is a Fredholm map with $i(A) \geq 0$ and $N : R^k \times X \to Y$ is a nonlinear G-equivariant map (cf. [Mi-6] for details).

2. Applications

A. **Nonlinear perturbations of Fredholm maps of nonnegative index.** Throughout this section we assume that $A : D(A) \subset X \to Y$ is a Fredholm map of nonnegative index, $X_0 = \ker A$, $\widetilde{Y} = R(A)$, and $\widetilde{X}$ and Y_0 are closed subspaces of X and Y, respectively, such that $X = X_0 \oplus \widetilde{X}$ and $Y = Y_0 \oplus \widetilde{Y}$ and $i(A) = \dim X_0 - \dim Y_0 = m$. Let $\Gamma_m = \{X_n, Y_n, Q_n\}$ be a projection scheme for (X, Y) with $X_0 \subset X_n$, $Q_n(\widetilde{Y}) \subset \widetilde{Y}$, and $\dim X_n - \dim Y_n = m$ for each n. Decompose X_0 as $X_0 = W \oplus Z$ with $\dim W = m$ and $\dim Z = \dim Y_0$. Let $Y_0 \subset Y_n$, $Q_n A x = Ax$ for $x \in X_n$, and $Q_n y \to y$ for each $y \in Y$. Let $P : X \to X_0$ and $Q : Y \to Y_0$ be linear projections onto X_0 and Y_0 respectively. For each $x \in X$, we have the unique decomposition $x = x_0 + x_1$, $x_0 \in X_0$, $x_1 \in \widetilde{X}$. Set $V = Z \oplus \widetilde{X}$, $X = W \oplus V$, and $V_n = X_n \cap V$.

If we try to establish continuation results of the type discussed in Section 1, we find that $\deg(QN, D \cap X_0, 0) = 0$ since $QN : \overline{D} \cap X_0 \to Y_0$ and $\dim Y_0 < \dim X_0$. To overcome this difficulty, we shall present now, in the context of A-proper maps, the A-essential mapping approach of the study of semilinear equations as developed by the author [Mi-4–6]. In particular, we shall apply it to complementing and G-equivariant maps. For compact nonlinearities, we refer to Nirenberg [Ni-1], Berger [Be], [MR].

Now, we shall apply Theorem 1.3 to homotopies of the form $H(t, x) = Ax + F(t, x)$, where $A : D(A) \subset X \to Y$ is as above. Let D be an open and bounded subset of X.

Theorem 2.1. *Let $A : D(A) \subset X \to Y$ be Fredholm of index $i(A) = m \geq 0$, and $F : [0, 1] \times X \to Y$ be nonlinear such that $H(t, x) = Ax + F(t, x)$ is a sectionally proper A-proper homotopy on $[0, 1] \times (\overline{D} \cap D(A))$ w.r.t. $\Gamma_m = \{X_n = W \oplus V_n, Y_n, Q_n\}$. Suppose that*

(i) *$A(x) + F(t, x) \neq f$ for $x \in \partial D \cap D(A)$, $t \in [0, 1)$,*

(ii) *$F(0, \cdot)(\overline{D}) \subset Y_0$,*

(iii) *$F(0, x) \neq t f_0$ for $x \in \partial D \cap X_0$, $t \in [0, 1]$,*

and either one of the following conditions holds:

(iv) *$m = 0$ and $\deg(F(0, \cdot), D \cap X_0, 0) \neq 0$.*

(v) *$m = 0$, X_0 and Y_0 are G-spaces, $F(0, \cdot) : \partial D \cap X_0 \to Y_0$ is continuous and G-equivariant with G satisfying condition* (G) *in X_0.*

(vi) *$m > 0$,* (i)–(iii) *hold with $D_0 = U \cap (Z \times \widetilde{X})$ for some open $U \subset X$ such that the set $\{(t, x) \in [0, 1] \times (D_0 \cap D(A)) \mid Ax + F(t, x) = f\}$ is relatively compact and $\deg(F(0, \cdot), D_0 \cap Z, 0) \neq 0$.*

(vii) *$m > 0$, $\overline{D} = \{x = x_0 + x_1 \mid \|x_0\| \leq r,\ \|x_1\| \leq R\}$ for some r, $R > 0$, and $F(0, \cdot) : \partial B(0, r) \cap X_0 \to Y_0 \backslash \{0\}$ has nontrivial stable homotopy.*

Then the equation $Ax + F(1, x) = f$ *is approximation-solvable in* $\overline{D} \cap D(A)$ *when* $m = 0$. *If* $m > 0$, *then there exists a connected subset* C *of* $S = \{x \in D(A) \mid Ax + F(1, x) = f\}$ *whose dimension is at least* m *at each point and either* C *is unbounded or* $C \cap \partial D \neq 0$. *Moreover, if* $D = X$ *and if* $\{x_n\} \subset S$ *is bounded whenever* $\{P_w x_n\}$ *is bounded, where* $P_w : X = W \oplus V \to W$ *is the projection onto* W, *then* C *covers* W *in the sense that* $P_w(C) = W$.

Proof. It is clear that conditions (ii) and (iii) imply condition (ii) of Theorem 1.3 for $H_t = A + F_t$. It remains only to check condition (iii) of this theorem. Since $X_0 \subset X_n$ and $Y_0 \subset Y_n$, we have that $Q_n H(0, \cdot) = A + F(0, \cdot)$ on $\overline{D} \cap X_n$. Define $F_n(t, x) = Ax + F(0, x_0 + tx_1)$ for $t \in [0, 1]$ and $x \in \overline{D} \cap X_n$, where $x = x_0 + x_1$ with $x_0 \in X_0$, $x_1 \in \widetilde{X}$. Since $\{x \in \overline{D} \cap D(A) \mid Ax + F(0, x) = 0\} = \{x = x_0 \mid x_0 \notin \partial D \cap X_0\}$ by (ii)–(iii), it follows that $F_n(t, x) \neq 0$ for $t \in [0, 1]$, $x \in \partial D \cap X_n$ and therefore

$$\deg(A + F(0, \cdot), D \cap X_n, 0) = \deg(A + F(0, P\cdot), D \cap X_n, 0).$$

Next, we have the decomposition $X_n = X_0 \times X_{1n}$, with $X_{1n} = X_n \cap \widetilde{X}$, and let $L_0 : Y_0 \to X_0$ be a linear homeomorphism. Then $L = (A^{-1}, L_0) : Y_{1n} \times Y_0 \to X_{1n} \times X_0$ is a linear homeomorphism and $LF_{n,0}(x_0, x_1) = (x_1, L_0 F(0, x_0))$. Hence, by the properties of Brouwer's degree we get

$$\begin{aligned} &\deg(LF_{n,0}, D \cap (X_{1n} \times X_0), 0) \\ &\quad = \deg(L, L^{-1}(D \cap (X_{1n} \times X_0)), 0) \deg(F_{n,0}, D \cap (X_0 \times X_{1n}), 0). \end{aligned}$$

Hence, if (iv) holds, then

$$\begin{aligned} \deg(A + F(0, P\cdot), D \cap X_n, 0) &= \pm \deg(LF_{n,0}, D \cap (X_0 \times X_{1n}), 0) \\ &= \pm \deg(F(0, \cdot), D \cap X_0, 0) \neq 0. \end{aligned}$$

If (v) holds, then it implies (iv) by the generalized Borsuk theorem [N-2]. Hence, condition (iii) of Theorem 1.3 is satisfied.

Let (vi) hold. By [FMP-1], we only need to show that $H(1, \cdot) = A + F(1, \cdot) - f : \overline{D}_0 \cap D(A) \subset W \times V \to Y$ is complemented by $P_w : X = W \times V \to W$ given by $P(w, v) = w$. By Proposition 3.1 in [FMP-1], this will be so if $\deg(Q_n H(1, \cdot), D_0 \cap V_n, 0) \neq 0$ for all large n. We have that

$$\deg(Q_n H(1, \cdot), D_0 \cap V_n, 0) = \deg(A + Q_n F(0, \cdot), D_0 \cap V_n, 0).$$

As above, we get, if $P_0 : X \to Z$ is a linear projection onto Z, then

$$\deg(A + F(0, \cdot), D_0 \cap V_n, 0) = \deg(A + F(0, P_0\cdot), D_0 \cap V_n, 0) \neq 0.$$

Let (vii) hold. Write $X_0 = R^k \times Z$ with $\dim Z = \dim Y_0$. By (ii)–(iii) we have that $H(0, x) \neq tf$ for all $x \in \partial D \cap D(A)$, $t \in [0, 1]$. As above, we have that $Q_n H_0 = A + F_0$ is homotopic to $A + F_0 P$ on $\overline{D} \cap X_n$ for each large n with $A + F_0 P : \partial D \cap X_n \to Y_n \backslash \{0\}$. As in the proof of Theorem 3.1 in [Mi-5], using (vii) we get the last map is essential on $\overline{D} \cap X_n$. Hence, H_0 is A-essential w.r.t. Γ_m on $\overline{D}$ and Theorem 1.3 applies. □

Corollary 2.1 [Mi-3,4,5,8]. *Let* $A + N : \overline{D} \cap D(A) \subset X \to Y$ *be sectionally proper and A-proper w.r.t.* Γ *with* N *bounded and nonlinear. Let* $f = f_0 + f_1 \in Y_0 \oplus \widetilde{Y}$ *and*

(i) *$A(x) + tNx \neq f$ and either $Ax \neq f_1$ or $QNx \neq f_0$ for $x \in \partial D \cap D(A)$, $t \in [0, 1)$.*

(ii) *$QNx \neq tf_0$ for $x \in \partial D \cap X_0$, $t \in [0, 1]$.*

(a) *Let $m = 0$ and either* $\deg(QN, D \cap X_0, 0) \neq 0$, *or X_0 and Y_0 are G-spaces and $QN : \partial D \cap X_0 \to Y_0$ is continuous and G-equivariant. Then* (3) *is approximation solvable.*

(b) *Let $i(A) = m > 0$ and $QN : \partial B(0, r) \cap X_0 \to Y_0 \backslash \{0\}$ have a nontrivial stable homotopy. Then there is a minimal connected subset Σ of $(A + N)^{-1}(f)$ whose dimension is at least m at each point.*

When $A + F_1$ is pseudo A-proper, we have

Theorem 2.2. *Let $A : D(A) \subset X \to Y$ be Fredholm with $i(A) = 0$, $F : [0, 1] \times X \to Y$ be nonlinear, $Q_n H_t = A + Q_n F_t$ be continuous on $\overline{D} \cap D(A) \cap X_n$ for $t \in [0, 1]$, $n \geq n_0$, and $H_1 : \overline{D} \cap D(A) \subset X \to Y$ be pseudo A-proper w.r.t. Γ_0 with $X_0 \subset X_n$, $Y_0 \subset Y_n$, and $Q_n(\widetilde{Y}) \subset \widetilde{Y}$. Assume*

(i) *$A(x) + Q_n F(t, x) \neq tQ_n f$ for $x \in \partial D \cap X_n$, $t \in [0, 1)$, $n \geq n_0$.*

(ii) *$F_0(\overline{D} \cap X_0) \subset Y_0$*

and either one of the following conditions holds:

(iii) $\deg(F_0, D \cap X_0, 0) \neq 0$.

(iv) *X_0 and Y_0 are G-spaces, $F_0 : \partial D \cap X_0 \to Y_0$ is continuous and G-equivariant for some group G satisfying condition* (G) *in X_0.*

Then the equation $Ax + F(1, x) = f$ is solvable in $\overline{D} \cap D(A)$.

Proof. As in the proof of Theorem 2.1, we have in either case that

$$\deg(A + F(0, \cdot), D \cap X_n, 0) = \deg(A + F(0, P\cdot), D \cap X_n, 0) \neq 0$$

for $n \geq n_0$. Hence, for each $n \geq n_0$, $\deg(A + Q_n F(1, \cdot), D \cap X_n, Q_n f) \neq 0$ and the conclusion follows from the pseudo A-properness of $A + F(1, \cdot)$. □

B. **k-parameter semilinear equations.** Let $A : X \to Y$ be a linear Fredholm map, $N : \overline{D} \subset R^k \times X \to Y$ a nonlinear map, and $m = i(A) + k \geq 0$ with $k \geq 0$. In this section we shall study the solvability and the covering dimension of the solution set of the k-parameter equations (2).

As before, we have the direct sums $X = X_0 \oplus \widetilde{X}$ and $Y = Y_0 \oplus \widetilde{Y}$ with $X_0 = \ker A$ and $\widetilde{Y} = R(A)$. Define $A_k : R^k \times X \to Y$ by $A_k(\lambda, x) = Ax$. Then (2) is equivalent to the equation

$$A_k(\lambda, x) + N(\lambda, x) = f. \tag{6}$$

Decompose $X_0 = W \oplus Z$ with $\dim Z = \dim Y_0$, $Z \subset X_0$ and set $V = Z \oplus \widetilde{X}$. Let $\Gamma_0 = \{V_n, Y_n, Q_n\}$ be an admissible scheme for (V, Y) and $\Gamma_m = \{R^k \oplus W \oplus V_n, Y_n, Q_n\}$, $m = \dim W + k$. Let $Q : Y \to Y_0$ be a linear projection onto Y_0, and for $(\lambda, x) = (\lambda, x_0 + x_1) \in R^k \oplus (X_0 \oplus \widetilde{X})$, we define $\|(\lambda, x)\|_1 = \max\{\|\lambda + x_0\|, \|x_1\|\}$. Let $\delta = \sup \|Q_n\|$. Suppose that

(7) For some $f_0 \in Y_0$ there are constants $M > 0$, $K > 0$, and $\rho \geq 0$ such that for $\|x_1\| \leq r$, $r > K$, $\|z\| \geq rM + \rho$, $z \in Z$,

$$QN(z + x_1) \neq tf_0.$$

(8) $S = (I - Q)N : R^k \times X \to Y$ is quasibounded, i.e.,

$$|S| = \limsup_{\|(\lambda, x)\|_1 \to \infty} \frac{\|S(\lambda, x)\|}{\|(\lambda, x)\|_1} < \infty$$

and $\delta|S|\max\{1, M\} < c$, where $\|Q_n A x_1\| \ge c\|x_1\|$ for $x_1 \in \widetilde{X}$.

Theorem 2.3. *Let* $m = i(A) + k \ge 0$, $N : R^k \times X \to Y$ *be a nonlinear map, and* $\Gamma_m = \{R^k \oplus W \oplus V_n, Y_n, Q_n\}$ *be a scheme for* $(R^k \times X, Y)$ *with* $Z \subset V_n$, $Y_0 \subset Y_n$, *and* $Q_n(\widetilde{Y}) \subset \widetilde{Y}$ *for* $n \ge 1$. *Suppose that* (7)–(8) *hold and*

(9) *For* $r > K$ *sufficiently large and* $\overline{D}_0 = \{z \in Z \mid \|z\| \le rM + \rho\}$, $\deg(QN, D_0, 0) \ne 0$.

(a) *If* $A + N$ *is pseudo* A*-proper w.r.t.* Γ_m *and* $m \ge 0$, *then* (2) *is solvable for each* $f \in f_0 \oplus \widetilde{Y}$.

(b) *If* $A + N$ *is sectionally proper and* A*-proper w.r.t.* Γ_m *and* $m > 0$, *then for each* $f \in f_0 \oplus \widetilde{Y}$ *there exists an unbounded connected closed subset* $C \subset \{(\lambda, x) \in R^k \times X | Ax + N(\lambda, x) = f\}$ *whose dimensional is at least* m *at each point and intersects* W.

(c) *Let* $m = 0$, $A+N$ *be pseudo* A*-proper w.r.t.* Γ_m *and, instead of* (9), *let* X_0 *and* Y_0 *be* G*-spaces and for* $r > K$ *sufficiently large,* $QN\colon \partial B(0, rM+\rho)\cap X_0 \to Y_0$ *be continuous and* G*-equivariant for some* G *satisfying condition* (G) *in* X_0. *Then* (3) *is solvable for each* $f \in f_0 \oplus \widetilde{Y}$.

Proof. Let $f \in f_0 + \widetilde{Y}$ be fixed and $N_f x = Nx - f$. We shall show that $\deg(Q_n(A + N_f)|V_n, V_n, 0) \ne 0$ for all large n. Arguing by contradiction and using arguments similar to [Mi-2], we get that there is an r_0 large and $n_0 \ge 1$ such that whenever $Q_n Ax + tQ_n(I - Q)Nx = tQ_n f_1$ for some $t \in [0, 1]$, $x = z + x_1 \in X_n$, $n \ge n_0$, with $\|z\| \le rM + \rho$ and $\|x_1\| = r$ for some r then $r \le r_0$. Define $H\colon [0, 1]\times V \to Y$ by $H(t, v) = Av + t(I-Q)Nv + QN(z+tx_1)$ and let $P\colon V \to Z$ be a linear projection onto Z. For $r > r_0$, set $\overline{D} = \{v = z + x_1 | \|x_1\| \le r,\ \|z\| \le rM + \rho,\ z \in Z\}$. Then $Q_n H(t, v) \ne tQ_n f$ for $v \notin D \cap V_n$, $t \in [0, 1]$, and $n \ge n_0$, and $\deg(Q_n(A + N_f)|V_n, V_n, 0) = \deg(Q_n(A + NP)|V_n, V_n, 0) \ne 0$ for $n \ge n_0$. Thus, by Proposition 3.1 in [FMP-1], $T = A + N_f\colon R^k \oplus W \oplus V \to Y$ is complemented by the projection $\widetilde{P}\colon R^k \oplus W \oplus V \to R^k \oplus W$ along V. Hence, (a) follows from the pseudo A-properness of $A + N$, while (b) follows from Theorem 1.3. Part (c) follows from Theorem 2.2. □

The above arguments combined with the arguments of the proof of Theorem 6 in [Mi-2] show that if $S = (I - Q)N$ has a sub(linear) growth (i.e. there are constants $\alpha \ge 0$, $\beta \ge 0$, and $\gamma \in [0, 1]$ with $\beta\delta < c$ if $\gamma = 1$, such that $\|S(\lambda, x)\| \le \alpha + \beta\|(\lambda, x)\|^\gamma$ for all $(\lambda, x) \in R^k \times X$), then condition (7) in Theorem 2.3 can be relaxed to

(10) For some $f_0 \in Y_0$ there are constants $M > 0$ and $K \ge 0$ such that for each $\|x_1\| \le r$, $r > K$, and each $\rho \ge r\rho(r)$ for some $\rho(r) \ge M$

$$QN(\rho z + \rho^\gamma x_1) \ne tf_0 \quad \text{for } z \in \partial B(0, 1) \cap Z,\ t \in [0, 1].$$

Remark 2.1. When $k = 0$, $i(A) > 0$, and N is continuous compact and uniformly bounded (i.e., $\|N\| \le C$ for all x), the solvability of $Ax+Nx = 0$ in Theorem 2.3 was given in Nirenberg [Ni-1] (cf. also [Cr]). When $k = i(A) = 0$, Theorem 2.3 was obtained by the author [Mi-2, 4]. For other special cases, see [Mi-6].

The next resonance conditions proved to be useful in studying the solvability of (3) with $i(A) = 0$ [N, F, Fi, Ma-2, Pe-2, Mi-2, 3, 4] and we shall now show their usefulness in the present situation. Define a continuous bilinear form $[\cdot, \cdot]: Y \times Z \to R$ such that $y \in \widetilde{Y}$ if and only if $[y, z] = 0$ for each $z \in Z$. As in Mawhin [Ma-2], if $\phi_1, \dots, \phi_n$ is a basis for Z, then the linear map $J: Y_0 \to Z$ given by $Jy_0 = \sum_1^n [y_0, \phi_i]\phi_i$ is an isomorphism, $[J^{-1}\phi_i, \phi_j] = \delta_{i,j}$, the Kronecker delta, and $[J^{-l}z, \phi_i] = c_i$ for $1 \le i \le n$, where $z = \sum_1^n c_i\phi_i$. If $(Y, (\,,\,))$ is a Hilbert space and $M: Z \to Y_0$ a linear isomorphism, then we can define $[y, z] = (y, Mz)$.

Let $f_0 \in Y_0$ be fixed and consider the following conditions.

(11) $N: R^k \times X \to Y$ is asymptotically zero (i.e.,

$$\|N(\lambda, x)\|/\|(\lambda, x)\| \to 0 \quad \text{as } \|(\lambda, x)\| \to \infty)$$

and either (i) $\liminf[Nv_n, z] < [f_0, z]$, or (ii) $\limsup[Nv_n, z] > [f_0, z]$ whenever $\{v_n\} \subset V$ is such that $\|v_n\| \to \infty$ and $v_n\|v_n\|^{-1} \to z \in Z$.

(12) N has a sublinear growth (i.e., for some $\alpha \ge 0$, $\beta \ge 0$, and $\gamma \in (0, 1)$,

$$\|N(\lambda, x)\| \le \alpha + \beta\|(\lambda, x)\|^\gamma \quad \text{for } (\lambda, x) \in R^k \times X)$$

and either (i)

$$\liminf[N(\rho_n z_n + \rho_n^\gamma v_n), z] < [f_0, z],$$

or (ii)

$$\limsup[N(\rho_n z_n + \rho_n^\gamma v_n), z] > [f_0, z]$$

whenever $\rho_n \to \infty$, $\{v_n\} \subset \widetilde{X}$ is bounded, and $\{z_n\} \subset Z$ is such that $z_n \to z \in \partial B(0, 1)$.

(13) N has a sublinear growth and either (i) $\liminf[Nv_n, u_n] < [f_0, z]$, or (ii) $\limsup[Nv_n, u_n] > [f_0, z]$ whenever $\{v_n\} \subset V$ is such that $\|P_n v_n\| \to \infty$, $\|(I-P)v_n\|/\|Pv_n\|^\gamma$ is bounded, and $u_n = Pv_n/\|Pv_n\| \to z \in Z$, where P is a linear projection of V onto Z.

(14) Suppose that X is a vector subspace of Y, $[\cdot, \cdot]: V \times Y \to R$ is a positive bilinear form such that if $\{v_n\} \subset V$ and $v_n \to v_0$, then $\liminf[v_n, x_0] \ge [v_0, x_0]$ and $Y = Z \oplus \widetilde{Y}$, where "$\oplus$" denotes orthogonal direct sum with respect to $[\cdot, \cdot]$. Let N have a linear growth, i.e, $\gamma = 1$ in (12), and either (i) $\liminf[Nv_n, u_n] < [f_0, z]$, or (ii) $\limsup[Nv_n, u_n] > [f_0, z]$ whenever $\{v_n\} \subset V$, $\|v_n\| \to \infty$, and $\limsup \|(I-P)v_n\|/\|Pv_n\| \le \beta(c^2 - \beta^2)^{-1/2}$, where $u_n = Pv_n/\|Pv_n\| \to z \in Z$ and c is the largest positive constant such that $c\|x\| \le \|Ax\|$ for $x \in R(A)$ and $\beta \in (0, c)$.

(15) (Antipodes condition) For a given $f_0 \in Y_0$ there exist constants $M \ge 0$, $K > 0$, and $\rho_0 \ge 0$ such that for each $\|x_1\| \le r$, $r > K$, $z \in \partial B(0, 1) \cap Z$, and $\rho \ge rM + \rho_0$

$$QN(\rho z + x_1) - f_0 \ne \mu QN(-\rho z - x_1) - \mu f_0 \quad \text{for } \mu \in [0, 1].$$

We shall also need that the scheme Γ_m for $(R^k \times X, Y)$ has the following **Property (P)** [Mi-4] (i) $Y_0 \subset Y_n$ and $Q_n(A+C)x = (A+C)x$ for $x \in V_n$, $n \ge 1$, where $C: V \to Y$ is linear and $B = A + C: V \to Y$ is bijective;

(ii) $[Q_n y, z] = [y, z]$ for each $y \in Y$, $z \in Z$.

Various schemes having Property (P) have been discussed in [Mi-3, 4]. For maps satisfying the above conditions we have

Theorem 2.4. *Let* $A: X \to Y$ *and* $N: R^k \times X \to Y$ *with* $m = i(A) + k \ge 0$, Γ_m *be as in Theorem* 2.3, *and either one of conditions* (11)–(14) *holds or* (8) *and* (15) *hold for some* $f_0 \in Y_0$. *Let* $f \in f_0 \oplus \widetilde{Y}$.

(a) *If* $A + tN$ *is sectionally proper and* A*-proper w.r.t.* Γ_m *for* $t \in [0, 1]$, $m > 0$ *and either one of conditions* (11)–(14) *holds, or if* $A + N$ *is* A*-proper and* (8) *and* (15) *hold, then there is an unbounded connected closed subset* K *of* $\{(\lambda, x) | Ax + N(\lambda, x) = f\}$ *whose dimension is at least* m *at each point and intersects* W.

(b) *If* $A + N$ *is pseudo* A*-proper w.r.t.* Γ_m *and* $m \geq 0$, *then* (2) *is solvable if either* $A|_V$ *is* A*-proper w.r.t.* Γ_m *having Property* (P) *and either one of conditions* (11)–(14) *holds, or* (8) *and* (15) *hold.*

Proof. Let $f \in f_0 + \widetilde{Y}$ be fixed, $N_f(\lambda, x) = N(\lambda, x) - f$, $C = \pm J^{-1}P \colon V \to Y_0$, and $H(t, v) = Av + tCv + (1-t)Nv$ for $t \in [0, 1]$ and $v \in V = Z \oplus \widetilde{X}$. Assume first that one of (11)–(14) holds. Arguing by contradiction as in [Mi-2] and using Property (P) we find an $R > 0$ such that for each $r \geq R$ there are γ and $n_0 \geq 1$ such that for $n \geq n_0$

$$\|Q_n H(t, v) - (1-t)Q_n f\| \geq \gamma \quad \text{for } v \in \partial B(0, r) \cap V_n,\ t \in [0, 1]. \tag{16}$$

Hence, $\deg(Q_n(A + N_f)|V_n, V_n, 0) \neq 0$ for each $n \geq n_0$. If $A + N$ is A-proper, it follows as before that $A + N$ is complemented by the projection $\widetilde{P}$ of $R^m \oplus W \oplus V$ onto $R^k \oplus W$ and so A-essential. Thus, (a) follows from Theorem 1.3. If $A + N$ is pseudo A-proper, then by Proposition 3.1 in [FMP-1] we get that $\deg(\widetilde{Q}_n(\widetilde{P}, A + N_f), B(0, r) \cap V_n, 0) \neq 0$ for each $n \geq n_0$ and therefore there exists a $v_n \in B(0, r) \cap V_n$ such that $Q_n A v_n + Q_n N v_n = Q_n f$. Hence, $Av + Nv = f$ for some v by the pseudo A-properness of $A + N$.

Now, assume that (8) and (15) hold. Let $f = f_0 + f_1$ be fixed and $R_0 > K$ such that $\|Sv\| < |S|\,\|v\|_1$ for each $\|v\|_1 \geq R_0$. Again, arguing by contradiction [Mi-2], we find an $R > R_0$ such that if, for some $\mu \in [0, 1]$, $n \geq n_0$, and $v \in V_n$ with $v = v_0 + v_1$, $\|v_0\| \leq rM + \rho_0$ and $\|v_1\| = r$ we have that

$$Q_n(I - Q)(Av - Nv - f) = \mu Q_n(I - Q)(-Av - N(-v) - f)$$

then $r \leq R$. For $k \geq R$, set $\overline{D} = \{v = v_0 + v_1 | v_0 \in Z,\ \|v_0\| \leq kM + \rho_0,\ \|v_1\| \leq k\}$. Let $H \colon [0, 1] \times \overline{D} \to Y$ be given by

$$H(t, v) = Av - 1/(1+t)N_f v - t/(1+t)N_f(-v)$$

with $N_f v = Nv - f$. Then $Q_n H(t, v) \neq tQ_n f$ for $v \in V_n \backslash \overline{D}$, $t \in [0, 1]$, and $n \geq n_0$. We get that $\deg(Q_n(A + N_f), D \cap V_n, 0) = \deg(Q_n H_1, D \cap V_n, 0) \neq 0$ for each $n \geq n_0$ since H_1 is an odd map. Hence, $A + N_f$ is complemented by the projection $\widetilde{P} \colon R^k \oplus W \oplus V \to R^k \oplus W$ if $A + N$ is A-proper. Then the conclusions of the theorem follow as for Theorem 2.3. □

When $k = i(A) = 0$, variants of (12) were first used by Necas [N], de Figueiredo [F], and Mawhin [Ma-2], while (11) and (13)–(14) were used by Fitzpatrick [Fi] in his study of (3) involving condensing maps and are improvements of the earlier conditions of Necas [N], Fucik [Fu-1], and Fucik-Kucera-Necas [FKN]. Still in this case, (11)–(12) and/or a variant of (13) were used in the study of (3) in the A-proper setting by the author [Mi-2–4] and Petryshyn [Pe-2]. Theorem 2.4 extends some of the results of these authors when $k = i(A) = 0$.

If $S = (I - Q)N$ has a (sub)linear growth, i.e. satisfies the inequality in (12) with $\delta\beta < c$, if $\gamma = 1$, then (15) in Theorem 2.4 can be relaxed to

(17) For some $f_0 \in Y_0$ there are constants $M > 0$ and $K \geq 0$ such that for each $\|x_1\| \leq r$, $r > K$, each $\rho \geq r\rho(r)$ for some $\rho(r) \geq M$, and $z \in \partial B(0, 1) \cap Z$

$$QN(\rho z + \rho^\gamma x_1) - f_0 \neq \mu QN(-\rho z - \rho^\gamma x_1) - \mu f_0 \quad \text{for } \mu \in [0, 1].$$

Remark 2.2. Inequalities (15) and (17) are valid in particular if, for all such ρ, z, x_1, γ, and μ and some linear isomorphism $J: Z \to Y_0^*$ (with $\gamma = 0$ in case of (15))

(18) $(QN(\rho z + \rho^\gamma x_1) - f_0 - \mu QN(-\rho z - \rho^\gamma x_1) + \mu f_0, Jz) \neq 0$.

It is clear that (18) is implied by either one of the inequalities

(19) $(QN(\rho z + \rho^\gamma x_1), Jz) < (f_0, Jz)$, or

(20) $(QN(\rho z + \rho^\gamma x_1), Jz) > (f_0, Jz)$.

When $i(A) = k = 0$ and (19) or (20) hold, the theorem was proved by the author [Mi-2] using the Leray-Schauder fixed point theorem in finite dimensions (see also Jarusek-Necas [JN]). Still in this case, the approximation solvability of (3) was proved in [Pe-2] under much stronger conditions on $H(t, x)$, which extends the earlier results of Mawhin [Ma-1] and Hetzer [H-1] when $Y = X$ and N is A-compact (condensing, respectively). The solvability of (3) with $i(A) = 0$ and S sublinear and N k-set-contractive, $k < 1$, was proved by Tarafdar [T] using different types of arguments.

Remark 2.3. Using the full force of Theorem 1.2 in [FMP-1], if we also know that the projection $\widetilde{P}$ of $W \oplus V$ onto W is proper, then in the results of this section with $m > 0$ and $A+N$ A-proper we can also assert that $\widetilde{P}(C) = W$ for the connected component of solutions of (3). This observation will be useful for our applications in subsection C.

C. **Applications to BVP's for differential equations.** Let $Q \subset R^n$ be an open and bounded region with smooth boundary. For $p \in (1, \infty)$ and some integers $k, m \geq 1$, we denote by $W_p^m = W_p^m(Q, R^k)$ the Sobolev space of R^k-valued functions. In this section we shall illustrate how some of the abstract results can be applied to the solvability of the BVP

$$(21) \qquad \begin{aligned} &\sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u(x) + F(x, u, Du, \ldots, D^m u) = 0 \quad \text{in } Q, \\ &B_j(u) = 0 \quad \text{on } \partial Q, \qquad j = 1, 2, \ldots, r. \end{aligned}$$

Assume that the linear part is elliptic. We shall consider BVP (21) with symmetries as well as when the associated operator can be complemented.

(A) Let us look at BVP (21) with symmetries. Suppose that $V = R^k$ is an orthogonal real representation of a compact Lie group G, whose action is $\{A_g | A_g = \text{orthogonal real } k \times k\text{-matrix}, \ g \in G\}$. Then $W_p^m(Q, V)$ has a structure of a Banach G-space with the linear action of G on it being given by $(T_g u)(x) = A_g(u(x))$, $x \in Q$, $g \in G$. We can identify $W_p^m(Q, V)$ with a tensor product of Banach G-spaces $W_p^m(Q, R) \otimes V$, and the left factor is a trivial G-space.

Suppose that for each $|\alpha| \leq m$, $A_\alpha(x)$ are sufficiently smooth maps from Q into $\mathrm{Hom}_R(V, V)$, the space of $k \times k$ real matrices, and the B_j's are boundary operators on ∂Q of order less than m. Denote by s_m the number of different derivatives D^α with $|\alpha| \leq m$ and $s_m' = s_m - s_{m-1}$. Define $X =$

$W_p^m(Q, V; \{B_j\}) = \{u \in W_p^m(Q, V) | B_j u = 0$ on ∂Q for $j = 1, \ldots, r\}$ and $Y = L_p(Q, V)$. Suppose that

(22) For each $|\alpha| \leq m$, the matrix-valued function $A_\alpha(x)$ is G-equivariant; i.e., $A_\alpha(x) A_g = A_g A_\alpha(x)$ for each $x \in Q$, $g \in G$.

(23) The boundary conditions $\{B_j(u) = 0\}$ are G-equivariant; i.e., $B_j(A_g u) = A_g B_j(u)$ for all $g \in G$, $j = 1, \ldots, r$.

(24) For each $v \in W_p^m$, the map $u \to F(x, u, \ldots, D^{m-1}u, D^m v)$ is continuous from W_p^{m-1} to L_p.

(25) For all $x \in Q, \eta \in R^{s_{m-1}}, \zeta, \zeta' \in R^{s'_m}$, and some $k > 0$ sufficiently small

$$|F(x, \eta, \zeta) - F(x, \eta, \zeta')| \leq k \sum_{|\alpha|=m} |\zeta_\alpha - \zeta'_\alpha|.$$

(26) The map $N: X \to Y$, given by $N(u)(x) = F(x, u(x), \ldots, D^m u(x))$ is G-equivariant; i.e., $N(T_g u)(x) = T_g N(u)(x)$ for all $g \in G$.

Using the classical results (Nirenberg [Ni-1]), one shows that $A: X \to Y$ defined by the linear part of (21), is a G-equivariant Fredholm map and its G-index $[\ker A] - [\operatorname{coker} A] \in RO^+(G)$, the semiring of isomorphism classes of real representations of G with direct sum and tensor product of representations as addition and multiplication, respectively of G. Then, Theorem 1.10 implies (see [Mi-6])

Theorem 2.5. *Suppose that conditions* (22)–(26) *hold and* $\operatorname{ind}_G(A) \in RO^+ + (G)$. *Then*

(a) *If* V *is such a representation that* G *satisfies condition* (G), *then BVP* (21) *has a solution* $u \in X$ *with* $\|u\| = r$ *for each* $r > 0$.

(b) *If* $G = S^1$ *and* $\operatorname{Fix}(S^1) = \{0\}$ *in* V, *then for each* $r > 0$ *and* $Z_r = \{u \in X | u$ *is a solution of* (21)$\}$, *the Fadell-Rabinowitz-index* $\operatorname{Ind}_C(Z_r) \geq \operatorname{ind} A$.

A special case of Theorem 2.5(a) is when F is odd, i.e., $G = Z_2$. It extends the results of Rabinowitz [R-2] with $G = Z_2$ and Marzantowicz [M-1] when F depends only on the derivatives up to order of $m - 1$, and the author [Mi-5].

In particular, let $Q = I = [0, 1]$, V be as above, and $C^k(V)$ be the space of V-valued C^k-functions on I with the norm

$$\|u\|_k = \sup_I (\|u(t)\|^2 + \cdots + \|u^{(k)}(t)\|^2)^{1/2}.$$

If G acts on $C^k(V)$ as above, then $Au = u^{(m)}(t): X = C^m(V) \to \widetilde{Y} = C(V)$ is a linear continuous G-equivariant epimorphism. Let W be a proper G-invariant subspace of U, the direct sum of m copies of V. If $i: \widetilde{Y} \to Y = W \oplus \widetilde{Y}$ is the G-equivariant inclusion, then $\widetilde{A} = iA: X \to Y$ is a G-equivariant Fredholm map with index $[V \oplus \cdots \oplus V] - [W] \in RO^+(G)$. Let the G-boundary conditions be given by a G-equivariant map $B: X \to W$. Suppose that $F: I \times U \to V$ is continuous and G-equivariant and for all $t \in I$, $\eta \in U \ominus V$, $\zeta, \zeta' \in V$, and a sufficiently small $k > 0$

$$|F(t, \eta, \zeta) - F(t, \eta, \zeta')| \leq k|\zeta - \zeta'|.$$

Then, under these assumptions, one shows that the conclusions of Theorem 2.5 hold in $C^m(V)$ for the BVP (which extends a result in [M-1])

$$u^{(m)}(t) = F(t, u(t), \ldots, u^{(m)}(t)), \qquad Bu = 0.$$

Note that if V and V' are two representations of G, of dimensions n and n', respectively, then a G-equivariant polynomial map $F: V \to V'$ is called a (V, V')-invariant. Thus, the invariant theory, which describes and classifies (V, V')-invariants, gives us examples of G-equivariant maps (see [We]).

(B) Let us now look at BVP (21) without symmetries when $k = 1$ and $p = 2$. Suppose that $Au = \sum_{|\alpha|\leq m} A_\alpha(x)D^\alpha u$ is elliptic and of index $m > 0$ from $X = W_2^m(Q, R; B_j)$ into $Y = L_2(Q, R)$, with B_j as before. As usual, if $X_0 = \ker A$ and $\widetilde{Y} = R(A)$, then $X = X_0 \oplus \widetilde{X}$ and $Y = Y_0 \oplus \widetilde{Y}$ with $\dim X_0 - \dim Y_0 = m$.

We now impose the following assumptions on F:

(27) $F: Q \times R^{s_m} \to R$ satisfies the Carathéodory conditions and there are $a, b \in L_2(Q)$ and $\gamma \in [0, 1]$ such that

$$|F(x, \xi)| \leq a(x) + b(x)|\xi|^\gamma \quad \text{for } x \in Q, \xi \in R^{s_m}.$$

(28) There exists $c > 0$ such that $\|Au\| \geq c\|u\|$ for $u \in \widetilde{X}$ and for $x \in Q$, $\eta \in R^{s_{m-1}}$, and $\zeta, \zeta' \in R^{s'_m}$

$$|F(x, \eta, \zeta) - F(x, \eta, \zeta')| \leq c \sum_{|\alpha|=m} |\zeta_\alpha - \zeta'_\alpha|.$$

(29) Suppose that $F(x, \xi) = G(x, \xi) + H(x, \xi)$ for $x \in Q, \xi \in R^{s_m}$ and
(i) $\lim_{s\to\pm\infty} G(x, s, \eta)/|s|^\gamma = g_\pm(x)$ uniformly in $\eta \in R^{s_m - 1}$,
(ii) there exist $c, d \in L_2(Q)$ and $\varepsilon > 0$ such that for $\gamma_1 < \min\{1, \gamma\}$

$$|H(x, \xi)| \leq \gamma_1[c(x) + d(x)|\xi|^{\gamma_1 - \varepsilon}] \quad \text{for } x \in Q, \xi \in R^{s_m},$$

(iii) there exists a subspace $Z \subset X_0$, $\dim Z = \dim Y_0$, and a linear bijection $J: Z \to Y_0$ such that $Ju = 0$ a.e. on $\{x | x \in Q, u(x) = 0\}$ and

$$\int_{u>0} g_+|u|^\gamma J(u)\,dx - \int_{u<0} g_-|u|^\gamma J(u)\,dx \neq \mu\left[\int_{u>0} g_-|u|^\gamma J(u)\,dx - \int_{u<0} g_+|u|^\gamma J(u)\,dx\right]$$

for $u \in \partial B(0, 1) \cap Z$ and $\mu \in [0, 1]$.

Theorem 2.6. *Under the above conditions on A and F there exists a connected closed subset C in X of solutions of BVP (21) whose dimension at each point is at least m and such that the projection of X onto $X_0 \ominus Z = W$ maps C onto W.*

Proof. Define $N: X \to Y$ by $Nu = F(x, u, Du, \ldots, D^{2m}u)$. Then there are constants a and b such that

$$\|Nu\| \leq a + b\|u\|^\gamma \quad \text{for } u \in X$$

and $A + N: X \to Y$ is A-proper with respect to a suitable scheme Γ as shown in [Mi-5]. Since it is continuous, it is proper on each bounded and closed set. Hence, the conclusions of the theorem follow from Theorem 2.4 and Remarks 2.2–2.3 provided we can show that (18) holds; i.e., there are constants $M \geq 0$, $K > 0$, and $\rho_0 \geq 0$ such that for each $v \in R(A)$ with $\|v\| \leq r$, $r > K$, $z \in \partial B(0, 1) \cap Z$, $\rho \geq rM + \rho_0$, and $\mu \in [0, 1]$

$$(N(\rho z + \rho^\gamma v) - \mu N(-\rho z - \rho^\gamma v), Jz) \neq 0$$

where $(\, , \,)$ is the L_2-inner product. This can be shown by arguing by contradiction and extending suitably the corresponding arguments in Tarafdar [T] for the case $i(A) = 0$, $\gamma < 1$, $H = 0$, and $F = F(x, u)$. □

When $i(A) = 0$, $\gamma < 1$, $H = 0$, $F = F(x, u)$ the existence assertion of Theorem 2.6 was obtained by Tarafdar [T]. For other resonance conditions with $i(A) > 0$, see [FMP-1, Mi-5, Ni-1].

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